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On coalescence of fermions on Riemann surfaces

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Abstract. We consider coalescing fermions on a Riemann surface and derive generalized determinant formulae, complementing some results of Constantinescu (1995 *Lett. Math. Phys.* **33** 195–206). Possible applications are indicated.

1. Introduction

Recall [1] that the $2N$-point function $\langle \prod_{i=1}^{N} b(z_i) \prod_{i=1}^{N} c(y_i) \rangle$ for the fermionic fields b, c on the Riemann sphere (i.e. Riemann surface of genus $g = 0$) is determined by the operator product expansions to $\prod_{i<j}(z_i - z_j) \prod_{i<j}(y_j - y_i) \prod_{i,j=1}^{N}(z_i - y_j)^{-1}$. Since the bc-system is a free system we may also use Wick's theorem to obtain $\det(\frac{1}{z_i - y_j})_{i,j=1}^{N}$, where $\langle b(z)c(y) \rangle = \frac{1}{z-y}$. Comparing the two expressions gives the Cauchy identity

$$\frac{\prod_{i<j}(z_i - z_j) \prod_{i<j}(y_j - y_i)}{\prod_{i,j=1}^{N}(z_i - y_j)} = \det\left(\frac{1}{z_i - y_j}\right)_{i,j=1}^{N}. \tag{1}$$

What happens when two (or more) fields coalesce, e.g. $z_i \to z_j$, was considered in [2, 3]. Considering the fields b, c as being charged (with opposite charge), this means that one wants to consider multiply charged fields where the entire system remains neutral; the limit $z_i \to w_j$ corresponds to the insertion of a current $j(z) = - : b(z)c(z):$ in w_j. In contrast to the case of the current where one has to extract the nonsingular terms, one has here to extract the nonvanishing terms in (1) carefully [3], obtaining so-called generalized Cauchy determinants. In the case $N = 2$ and $z_2 \to z_1$ one obtains the identity

$$\frac{(y_2 - y_1)}{(z_1 - y_1)^2(z_1 - y_2)^2} = \begin{vmatrix} \frac{1}{z_1 - y_1} & \frac{1}{z_1 - y_2} \\ \frac{1}{(z_1 - y_1)^2} & \frac{1}{(z_1 - y_2)^2} \end{vmatrix}. \tag{2}$$

Note that if we furthermore consider $y_2 \to y_1$ (corresponding to a system of two doubly charged fermions of opposite sign), we obtain

$$\frac{1}{(z_1 - y_1)^4} = \begin{vmatrix} \frac{1}{z_1 - y_1} & \frac{1}{(z_1 - y_1)^2} \\ \frac{1}{(z_1 - y_1)^2} & \frac{2}{(z_1 - y_1)^3} \end{vmatrix}. \tag{3}$$

The authors of [2] suggested considering the same situation in a higher genus; this is what we begin here. In the second section we recall very briefly the necessary facts of the bc-system in a higher genus, before we determine in the third section the determinant formulae. We indicate some possible applications in the last section.

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2. The *bc*-system

Let Σ_g be a Riemann surface of genus g and K its canonical bundle. The field b is a section of K^λ with $\lambda \in \frac{1}{2}\mathbb{Z}$, so c is a section of $K^{1-\lambda}$; more generally, we could consider twisted fermions where b is a section of a line bundle of degree $2\lambda(g-1)$. For half-integer λ we have to choose a theta-characteristic α with $\alpha^2 \simeq K$. We want to consider again the neutral case (the case where no zero modes exist), so that we have to choose by Riemann–Roch $\lambda = \frac{1}{2}$. For an even theta-characteristic there will be (generically) no zero modes, so this is the case we study. The two-point function for $g \geqslant 2$ is given by $\langle b(z)c(y)\rangle = \frac{\vartheta[\alpha](z-y)}{\vartheta[\alpha](0)E(z,y)} \equiv S_\alpha(z,y)$ (see e.g. [4,5]). Here we have used the prime form, which is a $-\frac{1}{2}$-form in each argument and behaves like $E(z,y) \sim z-y$ for z, y close and the theta-function $\vartheta[\alpha]$ corresponding to the chosen theta-characteristic α. Considering the four-point function in the two different ways yields Fay's trisecant identity [5,6]:

$$\frac{\vartheta[\alpha](z_1+z_2-y_1-y_2)E(z_1,z_2)E(y_2,y_1)}{\vartheta[\alpha](0)E(z_1,y_1)E(z_1,y_2)E(z_2,y_1)E(z_2,y_2)} = \begin{vmatrix} S_\alpha(z_1,y_1) & S_\alpha(z_1,y_2) \\ S_\alpha(z_2,y_1) & S_\alpha(z_2,y_2) \end{vmatrix}. \tag{4}$$

In the case $g = 1$ we obtain an identity of Frobenius [7]:

$$\frac{\sigma(z_1+z_2+w_1+w_2+\alpha)\sigma(\alpha)\sigma(z_1-z_2)\sigma(w_1-w_2)}{\sigma(z_1+w_1)\sigma(z_1+w_2)\sigma(z_2+w_1)\sigma(z_2+w_2)} = \det\left(\frac{\sigma(z_i+w_j+\alpha)}{\sigma(z_i+w_j)}\right)^2_{i,j=1}. \tag{5}$$

Here we have used the Weierstraß function σ which is closely related to the theta-function ϑ_1; cf [8,9]. For the applications we have to set $w_i = -y_i$.

3. A calculation

We will first consider the case of the four-point function in genus $g = 1$. Let $z_2 = z_1+\epsilon$; using $\sigma(-\epsilon) = -\sigma(\epsilon)$ and $\sigma(\epsilon) \sim \epsilon$ for small ϵ, we obtain for $\frac{1}{\epsilon}$ times the left-hand side of (5) in the limit $\epsilon \to 0$ the expression (recall that $w_i = -y_i$)

$$\sigma(\alpha)\sigma(2z_1-y_1-y_2+\alpha)\frac{\sigma(y_1-y_2)}{\sigma(z_1-y_1)^2\sigma(z_1-y_2)^2}. \tag{6}$$

Expanding furthermore $y_2 = y_1+\delta$, we obtain for $\delta^{-1}\cdot$ (6) in the limit $\delta \to 0$

$$-\sigma(\alpha)\frac{\sigma(2z_1-2y_1+\alpha)}{\sigma(z_1-y_1)^4}.$$

Let us now consider the right-hand side of (5). To shorten the notation, we introduce

$$Q_\alpha(z_i,y_j) := \frac{\sigma(z_i-y_j+\alpha)}{\sigma(z_i-y_j)}.$$

Denoting the derivative of Q_α with respect to the first (second) argument by D_z (D_y), we obtain for $z_2 = z_1+\epsilon \to z_1$ and $y_2 = y_1+\delta \to y_1$ the expansions

$$Q_\alpha(z_2,y_i) = Q_\alpha(z_1,y_i) + \epsilon\cdot D_z Q_\alpha(z_1,y_i) + \mathcal{O}(\epsilon^2)$$
$$Q_\alpha(z_1,y_2) = Q_\alpha(z_1,y_1) + \delta\cdot D_y Q_\alpha(z_1,y_1) + \mathcal{O}(\delta^2).$$

Expanding the right-hand side of (5) in ϵ yields as the term of first order

$$\epsilon\cdot\begin{vmatrix} Q_\alpha(z_1,y_1) & Q_\alpha(z_1,y_2) \\ D_zQ_\alpha(z_1,y_1) & D_zQ_\alpha(z_1,y_2) \end{vmatrix} \equiv \epsilon\cdot\begin{vmatrix} \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} & \frac{\sigma(z_1-y_2+\alpha)}{\sigma(z_1-y_2)} \\ D_z\frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} & D_z\frac{\sigma(z_1-y_2+\alpha)}{\sigma(z_1-y_2)} \end{vmatrix}. \tag{7}$$

Now, we consider $y_2 = y_1+\delta \to y_1$. Clearly, an analogous expansion will give us to first order in δ

$$\epsilon\delta\cdot\begin{vmatrix} Q_\alpha(z_1,y_1) & D_yQ_\alpha(z_1,y_1) \\ D_zQ_\alpha(z_1,y_1) & D_yD_zQ_\alpha(z_1,y_1) \end{vmatrix} \equiv \epsilon\delta\cdot\begin{vmatrix} \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} & D_y\frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} \\ D_z\frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} & D_yD_z\frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} \end{vmatrix}. \tag{8}$$

It remains to calculate the derivatives as explicitly as possible. Let

$$\psi_\alpha(z_i, y_j) := \frac{\sigma'}{\sigma}(z_i - y_j + \alpha) \qquad \psi(z_i, y_j) := \frac{\sigma'}{\sigma}(z_i - y_j) \qquad \Psi := \psi_\alpha - \psi. \tag{9}$$

Expanding the function $Q_\alpha(z_2, y_i)$ for $z_2 = z_1 + \epsilon$ and using that $\frac{a+b\epsilon+\mathcal{O}(\epsilon^2)}{c+d\epsilon+\mathcal{O}(\epsilon^2)} = \frac{a}{c} + \frac{bc-ad}{c^2}\epsilon + \mathcal{O}(\epsilon^2)$, we obtain $Q_\alpha(z_2, y_i) = Q_\alpha(z_1, y_i) + Q_\alpha(z_1, y_i) \cdot (\psi_\alpha(z_1, y_i) - \psi(z_1, y_i)) \cdot \epsilon + \mathcal{O}(\epsilon^2)$, thus identifying

$$D_z Q_\alpha(z_1, y_i) = Q_\alpha(z_1, y_i) \cdot (\psi_\alpha(z_1, y_i) - \psi(z_1, y_i)) = Q_\alpha(z_1, y_i) \cdot \Psi(z_1, y_i). \tag{10}$$

In a similar fashion (using this time that $\frac{a-b\epsilon+\mathcal{O}(\epsilon^2)}{c-d\epsilon+\mathcal{O}(\epsilon^2)} = \frac{a}{c} - \frac{bc-ad}{c^2}\epsilon + \mathcal{O}(\epsilon^2)$) one obtains

$$D_y Q_\alpha(z_1, y_i) = -Q_\alpha(z_1, y_i) \cdot (\psi_\alpha(z_1, y_i) - \psi(z_1, y_i)) = -Q_\alpha(z_1, y_i) \cdot \Psi(z_1, y_i) \tag{11}$$

(notice the relative sign). We may summarize the above observations as follows.

Proposition 1. *The 'generalized Frobenius determinant' arising in genus $g = 1$ from the four-point function $\langle b(z_1)b(z_2)c(y_1)c(y_2)\rangle$ in the limit $z_2 \to z_1$ is given by*

$$\sigma(\alpha)\frac{\sigma(2z_1 - y_1 - y_2 + \alpha)\sigma(y_1 - y_2)}{\sigma(z_1 - y_1)^2\sigma(z_1 - y_2)^2} = \begin{vmatrix} \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} & \frac{\sigma(z_1-y_2+\alpha)}{\sigma(z_1-y_2)} \\ \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} \cdot \Psi(z_1, y_1) & \frac{\sigma(z_1-y_2+\alpha)}{\sigma(z_1-y_2)} \cdot \Psi(z_1, y_2) \end{vmatrix}.$$

Considering furthermore the limit $y_2 \to y_1$, we obtain

$$\sigma(\alpha)\frac{\sigma(2z_1 - 2y_1 + \alpha)}{\sigma(z_1 - y_1)^4} = \begin{vmatrix} \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} & \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} \cdot \Psi \\ \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} \cdot \Psi & \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} \cdot [\Psi^2 - D_y\Psi] \end{vmatrix}$$

where we have suppressed the argument (z_1, y_1) of Ψ.

Some remarks are in order. It is now straightforward to generalize this to the case of $2N$-point functions, *as long as we consider only fermions of at most double charge*. For a general $2N$-point function $\langle \prod_{i=1}^N b(z_i) \prod_{i=1}^N c(y_i)\rangle$ the left-hand side of the analogous Frobenius identity is again easy to manipulate; on the right-hand side (i.e. the determinant) we have at worst the case where two derivatives have to be taken (one expansion in a z-variable and one expansion in a y-variable). This is exactly the case we considered above. Thus, it is possible (and straightforward, although a little tedious) to write down the corresponding 'generalized Frobenius determinant' following from a $2N$-point function, using the above formulas and the combinatorial structure given in [3]. It is of course possible to proceed *formally* (as in (7) and (8)) and consider the case of higher charged particles, involving higher and higher derivatives of the function Q_α—about which less and less seems to be known. This is in sharp contrast to the plane case ($g = 0$) considered in [3], where it is essential that $(\frac{1}{x})^{(n)} \propto \frac{1}{x^{n+1}}$.

Note that we may write the first equation of the proposition with the help of the addition theorem (example 2 on p 451 in [8]),

$$\frac{\sigma'}{\sigma}(u + v) = \frac{\sigma'}{\sigma}(u) + \frac{\sigma'}{\sigma}(v) + \frac{1}{2}\underbrace{\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}}_{=:\mathfrak{S}(u,v)}$$

in the following form:

$$\frac{\sigma(\alpha)\sigma(2z_1 - y_1 - y_2 + \alpha)\sigma(y_1 - y_2)}{\sigma(z_1 - y_1)^2\sigma(z_1 - y_2)^2} = \frac{1}{2}\begin{vmatrix} \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)} & \frac{\sigma(z_1-y_2+\alpha)}{\sigma(z_1-y_2)} \\ \frac{\sigma(z_1-y_1+\alpha)}{\sigma(z_1-y_1)}\mathfrak{S}(z_1, y_1) & \frac{\sigma(z_1-y_2+\alpha)}{\sigma(z_1-y_2)}\mathfrak{S}(z_1, y_2) \end{vmatrix}.$$

Let us check that this reduces to (2) in the case where all variables are close together, i.e. to the case $N = 2$ of [2, 3]. This has to be expected, since the particles should no longer feel the global (nontrivial) topology. Because of $\sigma(u) \sim u$ for small u the left-hand side is given roughly by $\frac{\sigma(\alpha)^2(y_1-y_2)}{(z_1-y_1)^2(z_1-y_2)^2}$, which is $-\sigma(\alpha)^2$ times the left-hand side of (2). Since $\wp(u) \sim \frac{1}{u^2}$, hence $\wp'(u) \sim -\frac{2}{u^3}$, we find for the right-hand side

$$\sim \frac{1}{2}\begin{vmatrix} \frac{\sigma(\alpha)}{z_1-y_1} & \frac{\sigma(\alpha)}{z_1-y_2} \\ -\frac{2\sigma(\alpha)}{(z_1-y_1)^2} & -\frac{2\sigma(\alpha)}{(z_1-y_2)^2} \end{vmatrix} = -\sigma(\alpha)^2 \begin{vmatrix} \frac{1}{z_1-y_1} & \frac{1}{z_1-y_2} \\ \frac{1}{(z_1-y_1)^2} & \frac{1}{(z_1-y_2)^2} \end{vmatrix}$$

which is nothing but the right-hand side of (2) mulitplied by $-\sigma(\alpha)^2$. Thus, in this 'limit' we obtain indeed the generalized Cauchy identity (2). Similarly, the second equation of proposition 1 'reduces' to (3).

In the case $g \geqslant 2$ we have to perform the same steps: setting $z_2 = z_1 + \epsilon$ and expanding in ϵ (and setting $y_2 = y_1 + \delta$ later); for these calculations compare also [10]. Let us again consider the case $N = 2$. The limits for the left-hand side of (4) are again very easy to determine. In analogy to the above we expand the Szegö kernel for $z_2 = z_1 + \epsilon \to z_1$ formally as $S_\alpha(z_2, y_i) = S_\alpha(z_1, y_i) + \epsilon \cdot D_z S_\alpha(z_1, y_i) + \mathcal{O}(\epsilon^2)$; here we have denoted the derivative with respect to the first argument by D_z (the derivative with respect to the second argument will be denoted by D_y). Manipulating the right-hand side as in the case of $g = 1$ we obtain in the limit $z_2 \to z_1$

$$\frac{\vartheta[\alpha](2z_1 - y_1 - y_2)E(y_2, y_1)}{\vartheta[\alpha](0)E(z_1, y_1)^2 E(z_1, y_2)^2} = \begin{vmatrix} S_\alpha(z_1, y_1) & S_\alpha(z_1, y_2) \\ D_z S_\alpha(z_1, y_1) & D_z S_\alpha(z_1, y_2) \end{vmatrix}. \tag{12}$$

Moreover, letting $y_2 \to y_1$ yields

$$-\frac{\vartheta[\alpha](2z_1 - 2y_1)}{\vartheta[\alpha](0)E(z_1, y_1)^4} = \begin{vmatrix} S_\alpha(z_1, y_1) & D_y S_\alpha(z_1, y_1) \\ D_z S_\alpha(z_1, y_1) & D_y D_z S_\alpha(z_1, y_1) \end{vmatrix}. \tag{13}$$

As in the case $g = 1$ it is again possible to proceed formally and consider the coalescence of more than two fields, involving higher derivatives such as $D_z^{n_1} D_y^{n_2} S_\alpha(z_i, y_k)$. Explicit expressions for these derivatives seem to exist only for $n_1 + n_2 \leqslant 2$, cf [9, 10]. Let us introduce the following functions in analogy to (9):

$$\phi_{x,\alpha}(z_1, y_i) := \frac{D_x \vartheta[\alpha](z_1 - y_i)}{\vartheta[\alpha](z_1 - y_i)} \qquad \phi_x(z_1, y_i) := \frac{D_x E(z_1, y_i)}{E(z_1, y_i)} \qquad \Phi_x := \phi_{x,\alpha} - \phi_x \tag{14}$$

where x stands for z or y. Using these functions and the explicit form of the Szegö kernel given in section 2, we obtain in analogy to (10) and (11)

$$D_z S_\alpha(z_1, y_i) = S_\alpha(z_1, y_i) \cdot (\phi_{z,\alpha}(z_1, y_i) - \phi_z(z_1, y_i)) = S_\alpha(z_1, y_i) \cdot \Phi_z(z_1, y_i)$$
$$D_y S_\alpha(z_1, y_i) = S_\alpha(z_1, y_i) \cdot (\phi_{y,\alpha}(z_1, y_i) - \phi_y(z_1, y_i)) = S_\alpha(z_1, y_i) \cdot \Phi_y(z_1, y_i).$$

Although the Szegö kernel is antisymmetric in the arguments z, y (we have chosen α to be an even theta-characteristic) it is not a function of the difference $z - y$, so we do not obtain $D_y S_\alpha(z_1, y_1) = -D_z S_\alpha(z_1, y_1)$, in contrast to the case of genus one (see (10) and (11)). Thus, we have in close analogy to proposition 1 the following.

Proposition 2. *The 'generalized Fay determinant' arising in genus $g \geqslant 2$ from the four-point function $\langle b(z_1)b(z_2)c(y_1)c(y_2)\rangle$ in the limit $z_2 \to z_1$ is given by*

$$\frac{\vartheta[\alpha](2z_1 - y_1 - y_2)E(y_2, y_1)}{\vartheta[\alpha](0)E(z_1, y_1)^2 E(z_1, y_2)^2} = \begin{vmatrix} S_\alpha(z_1, y_1) & S_\alpha(z_1, y_2) \\ S_\alpha(z_1, y_1) \cdot \Phi_z(z_1, y_1) & S_\alpha(z_1, y_2) \cdot \Phi_z(z_1, y_2) \end{vmatrix}.$$

Considering furthermore the limit $y_2 \to y_1$ yields

$$\frac{\vartheta[\alpha](2z_1 - 2y_1)}{\vartheta[\alpha](0)E(z_1, y_1)^4} = -\begin{vmatrix} S_\alpha(z_1, y_1) & S_\alpha(z_1, y_1) \cdot \Phi_y \\ S_\alpha(z_1, y_1) \cdot \Phi_z & S_\alpha(z_1, y_1) \cdot [\Phi_y \Phi_z + D_y \Phi_z] \end{vmatrix}$$

where we have again suppressed the argument (z_1, y_1) of Φ_x.

The second equation differs slightly from the second equation of proposition 1; if we had $\Phi_y = -\Phi_z \equiv -\Phi$, then there would be a perfect analogy. Since $D_y\Phi_z = D_z\Phi_y$ the order of the coalescing particles is arbitrary. Note that we can again generalize this result at once to the case of $2N$-point functions, as long as we consider only doubly charged fermions. It is again possible to check that the first equation (second equation) 'reduces' to (2) ((3)) when all variables come close together.

Associating with the field b (c) the charge $q = 1$ ($q = -1$), we can write $\langle b(z)c(w)\rangle \equiv \langle b(x_1)c(x_2)\rangle = \frac{\vartheta[\alpha](x_1-x_2)}{\vartheta[\alpha](0)E(x_1,x_2)} = \frac{\vartheta[\alpha](q_1x_1+q_2x_2)}{\vartheta[\alpha](0)}E^{q_1q_2}(x_1, x_2)$, where q_i is the charge of the field sitting in x_i. Similarly, we can write the four-point function as

$$\langle b(x_1)b(x_2)c(x_3)c(x_4)\rangle = \frac{\vartheta[\alpha](\sum_{i=1}^{4} q_i x_i)}{\vartheta[\alpha](0)} \prod_{1\leqslant i<j\leqslant 4} E^{q_iq_j}(x_i, x_j) \tag{15}$$

where $q_i = \pm 1$. The Abelian bosonization [5, 11] shows that $b(x_i)$ ($c(x_j)$) corresponds to the chiral vertex $V_{+1}(x_i)$ ($V_{-1}(x_j)$), where $V_q(z) = :\mathrm{e}^{\mathrm{i}q\phi(z)}:$ for $q \in \mathbb{Z}$ and ϕ is a circle-valued bosonic scalar field with action $S \sim \int_{\Sigma_g} \partial\phi\bar\partial\phi\,\mathrm{d(vol)}$. We may thus write the left-hand side of (15) simply as $\langle\prod_{i=1}^{4} V_{q_i}(x_i)\rangle$. The operator product expansion $b(z)b(w) = b(z)b(z) + (w-z)b(z)\partial b(z) + \mathcal{O}((w-z)^2)$ yields that

$$B(z) := \lim_{w\to z} \frac{b(z)b(w)}{z-w} = -b(z)\partial b(z) \tag{16}$$

and similarly for $C(w)$. As in the case of $g = 0$ we have the correspondence $B(z) \leftrightarrow V_{+2}(z)$ and $C(w) \leftrightarrow V_{-2}(w)$. The left-hand side of the second equation of proposition 2 is thus given by $\langle B(z_1)C(y_1)\rangle = \langle V_{+2}(z_1)V_{-2}(y_1)\rangle$, or, in the more familiar notation of (15), by

$$\langle B(x_1)C(x_2)\rangle = \langle V_{\tilde q_1}(x_1)V_{\tilde q_2}(x_2)\rangle = \frac{\vartheta[\alpha](\sum_{i=1}^{2}\tilde q_i x_i)}{\vartheta[\alpha](0)} E^{\tilde q_1 \tilde q_2}(x_1, x_2)$$

where $\tilde q_1 = q_1 + q_2 = 2$ and $\tilde q_2 = q_3 + q_4 = -2$. Thus, the proposition shows that one can express the correlation functions involving higher charged vertices as determinants of vertices with lower charges (recall that $S_\alpha(z_1, y_1) = \langle V_{+1}(z_1)V_{-1}(y_1)\rangle$). Of course, this can be extended to general $2N$-point functions. Note that the neutrality condition $\sum_{i=1}^{2N} q_i = 0$ will always be satisfied.

The formula (16) shows that the fields $B(z)$ ($C(w)$) live on the first infinitesimal neighbourhood of the corresponding diagonal (we take a derivative); as the formulas for the plane case (i.e., $g = 0$) in [2] show, the fields of charge n involve derivatives up to order n, so they live on the nth infinitesimal neighbourhood.

4. Discussion

We have seen that it is possible to consider explicitly the case of coalescing fermions on Riemann surfaces of genus $g \geqslant 1$ in a straightforward fashion, as long as at most two of them flow together. This is only a first step, but for considering more general situations one has to have detailed control over derivatives of prime forms and theta-functions. The fields b and c are sections of a line bundle over the Riemann surface Σ_g. In the intrinsic interpretation the current $j(z) = -:b(z)c(z):$ is a section of a line bundle over the first infinitesimal neighbourhood of the diagonal in $\Sigma_g \times \Sigma_g$, cf [12]. Considering correlation functions with insertions of currents yields various corollaries to Fay's trisecant identity [12]. One may hope that the intrinsic interpretation of the limits considered above (and the more general ones with more than two fields flowing together) yields some new identities.

The generalized Cauchy determinants were used in [2, 3] to consider the (asymmetric) Coulomb gas in the complex plane; for a discussion of the Coulomb gas see [13]. Since the Coulomb gas is studied on Riemann surfaces as well (see e.g. [14, 15]), one should be able to transfer some of the conclusions of [2, 3] about the asymmetric Coulomb gas to higher-genus surfaces. In particular, this might yield some new insights into the quantum Hall effect on Riemann surfaces; cf [16, 17].

As stressed in [2] (and indicated above), one can consider the generalized Cauchy determinants as being associated with a kind of generalized bosonization. A general framework for bosonization in higher genus is given in [18] (see also [5, 11]); we expect that the generalized Frobenius (Fay) determinants will appear in concrete models for the analogous generalized bosonizations in higher genus.

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